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Centrifugal instability of time-dependent flows. Part 1. Inviscid, periodic flows

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An investigation is made of the stability of time-periodic azimuthal flows between coaxial, circular cylinders. The disturbance equations are linearized and consideration is limited to the effects of axisymmetric disturbances in a fluid with zero viscosity. It is found convenient to examine separately the cases of flows with zero time-mean and non-zero time-mean respectively. Some remarks are made concerning the definition of stability in relation to such flows and their general stability characteristics are evaluated and discussed.

1. Introduction

Although the literature in the field of hydrodynamic stability is a vast one, a surprisingly small proportion of it has been devoted to examining the stability of flows which vary with time. Since such flows occur abundantly in nature, it is presumably because of the mathematical difficulties involved that their study has been so neglected. In recent years, however, some serious attention has been given to the problem. For example, Greenspan & Benney (1963) and Drazin (1967) have considered broken-line shear profiles under various conditions of time-dependence, and Kelly (1965) has analysed an unsteady Kelvin–Helmholtz flow; while Shen (1961) and Conrad & Criminale (1965*a*, *b*) have discussed several plane and circumferential viscous flows with time-variation.

In spite of these and other developments, it must be said that the effects of time-dependence on stability characteristics are only partially understood. It is the object of this paper to contribute something to that understanding by examining in detail a particular class of flows: those which are periodic in time, which are azimuthal in the space between coaxial, circular cylinders and which have zero viscosity. This last assumption will naturally impose a serious constraint on the validity of the conclusions from a physical viewpoint, but it would seem nevertheless useful to analyse the simpler, inviscid problem as a preliminary to the viscous-flow problem which will be considered in a subsequent paper.

It is necessary at the outset to specify carefully what is understood by stability and instability in this context. When the basic flow is steady the disturbance, analysed into normal modes, has exponential time-dependence which can be extracted from the (linearized) perturbation equations. Instability, defined as the growth of a disturbance, is then settled by the nature of the exponent of the exponential. But when the basic flow is unsteady this approach does not apply.

In the first place the disturbance is no longer expressible as a simple function of time and the mathematical problem is consequently more formidable. Secondly, growth of a disturbance is, while necessary, not always a sufficient condition for instability. For when the flow itself is changing, what is important is the *relative* growth of disturbance and basic flow, since it is this which determines whether the character of the flow changes as a result of interaction with the disturbance.

A definition which incorporates this relative concept has been introduced by Shen (1961): a flow is unstable *at the instant* $t = t_0$ if the ratio of the disturbance energy to the basic-flow energy is tending to increase at that instant, and stable if it is tending to decrease. This definition has the defect of being quasi-steady; when applied to a particular flow characterized by a parameter, such as the Reynolds number [as has been done by Conrad & Criminale (1965*a*, *b*)] it will yield a critical value for the parameter which is different at each instant of time. If extended over an interval of time, therefore, it can only give a lower bound of critical values, or, equivalently, a sufficient condition for stability. Moreover, a quasi-steady criterion may not provide any useful information at all about the flow, as, for example, when the energy-ratio is increasing during part of a cycle of a periodic flow and decreasing during another part.

Periodic flows are somewhat special, and often simpler than the accelerating and retarding flows which were the main focus of Shen's (1961) study. Since they are bounded for all time, there is in general little difference between a criterion based on the growth of the disturbance itself and one which considers the disturbance-flow ratio. As suggested earlier, quasi-steady criteria are not very helpful for such flows, but, on the other hand, the asymptotic behaviour at $t \to \infty$ of the disturbance is usually an adequate test of stability (Kelly 1965; Drazin 1967). For inviscid fluids, moreover, since stability generally means neutral stability, boundedness at infinite time is equivalent to periodicity of the disturbance.

However, as we shall see below, these tests are not invariably satisfactory. It is well to recall that our mathematical treatment is restricted to an analysis of *linearized* disturbance equations and that consequently a stable disturbance must not only tend to zero (or be bounded for inviscid fluids) as $t \to \infty$, but it must also remain sufficiently small for linearization to be valid throughout all time. This in fact is the basic criterion of instability: an infinitesimal disturbance which grows to a finite size is unstable. In a linearized system it grows beyond the limits of the linearization. How this criterion is applied in mathematical analysis can vary depending on the nature of the basic flow. It is therefore necessary to apply a boundedness or periodicity test with some care, since even a purely periodic disturbance may have to be regarded as unstable in the sense that it leaves the régime of linearization during part of its cycle.

These ideas are illustrated in § 3 below in application to a simple model flow. For more general flows, it is convenient to examine separately profiles which have zero mean and non-zero mean with respect to time. This is done in §§ 4 and 5 respectively and a general discussion follows in § 6.

2. Formulation

The motion of the fluid is described by Euler's equations and the equation of continuity which are, in the usual notation,

$$\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p, \tag{2.1}$$

$$\operatorname{div}\mathbf{q} = 0. \tag{2.2}$$

The fluid is taken to be confined between coaxial circular cylinders of infinite length and of radii R_1, R_2, and the axis $r = 0$ of the cylindrical polar co-ordinate system (r, θ, z) is coincident with the common axis of the cylinders.

The basic flow whose stability is to be examined is a purely azimuthal motion described by the velocity vector

$$\bar{\mathbf{q}} = [0, \bar{v}(r,t), 0], \tag{2.3}$$

where $\bar{v}(r,t)$ has the time-dependence

$$\bar{v}(r,t) = \bar{v}\left(r, t+\frac{2\pi}{\omega}\right), \quad \text{all } t. \tag{2.4}$$

The vector $\bar{\mathbf{q}}$ defined by (2.3) satisfies the continuity equation, but may not necessarily be a solution of the equations of motion (2.1); if it is not, we shall suppose that it is an approximation to a solution of the full Navier–Stokes equations. This procedure may be justified on the grounds that our objective is to extract basic qualitative features of such flows in the absence of viscosity, rather than to calculate quantitative results for particular velocity profiles. (In fact the most general inviscid solution of the type stipulated, which requires both a radial and an azimuthal pressure gradient to maintain it, is

$$\bar{v}(r,t) = (1/r)\,F(t)+G(r),$$

where F, G are arbitrary functions.)

Introduce now a disturbance velocity vector defined by

$$\mathbf{q} = [0, \bar{v}(r,t), 0]+[u^*(r,z,t),\ v^*(r,z,t), w^*(r,z,t)], \tag{2.5}$$

in which the class of disturbances is restricted to those with axial symmetry. Although it is not *a priori* clear that axi-symmetric disturbances would be the critical ones for a general flow it is permissible to examine their effects alone in the qualitative, inviscid analysis which is proposed. It is assumed that the perturbation velocities u^*, v^*, w^* are sufficiently small to permit linearization of (2.1) and further that a normal mode analysis of the disturbance is valid. Then we set

$$u^* = u(r,t)\cos kz, \quad v^* = v(r,t)\cos kz, \quad w^* = w(r,t)\sin kz, \tag{2.6}$$

substitute into (2.1) and (2.2) and linearize. After elimination of w and the pressure, the system reduces to a pair of equations for u and v, which may be written

$$(DD_* - k^2)\frac{\partial u}{\partial t}+2k^2\left(\frac{\bar{v}}{r}\right)v = 0, \tag{2.7}$$

$$\frac{\partial v}{\partial t}+(D_*\bar{v})\,u = 0, \tag{2.8}$$

where $$D \equiv \frac{\partial}{\partial r} \quad \text{and} \quad D_* \equiv \frac{\partial}{\partial r}+\frac{1}{r}.$$

As might have been expected, (2.7) and (2.8) are identical with the stability equations for a steady basic flow under the same circumstances, except of course that $\bar{v}$ is a function of time. Consequently we have an eigenvalue problem for a partial differential system, the boundary conditions being as usual

$$u = 0 \quad \text{on} \quad r = R_1 \quad \text{and} \quad r = R_2. \tag{2.9}$$

3. Rigid-body oscillations

We consider first the stability of the almost trivial velocity profile

$$\bar{v}(r,t) = V(r)\,F(\omega t), \tag{3.1}$$

where $V(r)$ is an arbitrary differentiable function in $R_1 \leqslant r \leqslant R_2$, and F is periodic and has *zero mean*. This flow has the characteristic feature that the phase of the oscillation is independent of radial distance.

The transformation

$$\tau = \frac{1}{\omega}\int^{\omega t} F(s)\,ds \tag{3.2}$$

converts the disturbance equations (2.7) and (2.8) into

$$(DD_* - k^2)\frac{\partial u}{\partial \tau}+2k^2\left(\frac{V}{r}\right)v = 0 \quad \text{and} \quad \frac{\partial v}{\partial \tau}+(D_* V)\,u = 0, \tag{3.3}$$

where now $u = u\,(r,\tau), v = v(r,\tau)$. These equations, with time-independent coefficients, are of course precisely those which govern the stability of the steady flow $V(r)$. They admit solutions

$$u = e^{\gamma\tau}u_1(r), \quad v = e^{\gamma\tau}v_1(r), \tag{3.4}$$

where the nature of the eigenvalue γ is established from the classical Rayleigh criterion. If the discriminant,

$$\frac{V}{r}(D_* V),$$

is positive everywhere in (R_1, R_2), then all the γ are pure imaginary and occur in pairs

$$\gamma = \pm i\gamma_i.$$

If the discriminant is negative everywhere in the interval, then there is at least one pair of real γ which have the form

$$\gamma = \pm\gamma_r.$$

In view of (3.2) the time-dependence of the disturbances has in the former case the functional form

$$\begin{matrix}\cos\\ \sin\end{matrix}\left\{\frac{\gamma_i}{\omega}\int^{\omega t}F(s)\,ds\right\}, \tag{3.5}$$

and in the latter case

$$\exp\left\{\frac{\gamma_r}{\omega}\int^{\omega t} F(s)\,ds\right\}, \tag{3.6}$$

the γ_i and γ_r being in principle determined once $V(r)$ is prescribed.

When (3.5) applies the disturbance is bounded for all t and its magnitude remains $O(1)$ compared with any initial value; the flow is therefore stable. When (3.6) is relevant the disturbance is again bounded for all t and in this sense is also stable. It follows that if boundedness (or periodicity) is the criterion of stability, then all rigid-body oscillations (3.1) must be regarded as stable.

However, this conclusion is not altogether acceptable from a physical viewpoint. It is clear that if the oscillations are sufficiently slow, the stability behaviour will be quasi-steady; that is, the flow will be stable or unstable as $V(r)$ is stable or unstable. This apparent contradiction can be resolved by observing that when $V(r)$ is unstable, (3.6) indicates that the magnitude of the disturbance is increased $\exp(\gamma_r/\omega)$-fold during the growth part of its cycle. Since γ_r is in general $O(1)$, this factor is large enough, when $\omega \ll 1$, for the disturbance to leave the linearized régime†.

It follows that at very low frequencies boundedness of the disturbance is not an adequate criterion of stability, and in this range quasi-steady considerations apply. Outside the low-frequency range, the flow (3.1) is stable for all $V(r)$.

These considerations can immediately be extended to rigid-body oscillations with non-zero mean, as, for example,

$$\bar{v}(r,t) = V(r), \quad F(\omega t) = V(r)\,[1+G(\omega t)], \tag{3.7}$$

where now G is periodic with zero mean. The transformation (3.2) is still applicable, and the time-behaviour is again given by (3.5) or (3.6) which now have the form

$$\begin{matrix}\cos\\ \sin\end{matrix}\left\{\gamma_i t+\frac{\gamma_i}{\omega}\int^{\omega t} G(s)\,ds\right\} \tag{3.8}$$

or

$$\exp\left\{\gamma_r t+\frac{\gamma_r}{\omega}\int^{\omega t} G(s)\,ds\right\}. \tag{3.9}$$

Hence when $V(r)$ is stable, (3.7) is stable for all frequencies. When $V(r)$ is unstable, (3.7) is unstable and this too holds for the whole frequency range. However, the growth rate of the disturbance is a function of time, and when $\omega \ll 1$ the rate of change of this function is large. This fact would have a considerable effect on the further development of the disturbance, i.e. into the non-linear régime, but nothing more can be said within the confines of linearized theory.

4. Flows with zero mean

As a sequel to the above special results, it would seem natural to consider next the effects of phase differential in the radial direction, while retaining the zero-mean property. In the first instance we take the departure from rigid-body oscillation to be *small.*

† These remarks apply to a disturbance whose initial amplitude is fixed as $\omega \to 0$. It would of course always be possible to find a disturbance sufficiently small for linear analysis to be valid when $\omega \ll 1$.

A fairly general profile of this type would have the form

$$\bar{v}(r,t) = V[r, \omega t + \epsilon(r)], \tag{4.1}$$

with

$$\int_t^{t+2\pi/\omega} V\,dt = 0 \tag{4.2}$$

and $|\epsilon| \ll 1$, but in place of (4.1) it is sufficient to consider

$$\bar{v}(r,t) = V_1(r)\cos\omega t + \epsilon V_2(r)\sin\omega t, \tag{4.3}$$

with ϵ now a small constant. This, while somewhat less general than (4.1), retains the required characteristics.

It is convenient at this stage to non-dimensionalize all quantities. Set

$$\left.\begin{aligned} x = r/R_2, \quad \eta = R_1/R_2, \quad \tau = \omega t, \\ a = kR_2 \quad \text{and} \quad V_1(r), V_2(r) \to \Omega R_2[V_1(x), V_2(x)], \end{aligned}\right\} \tag{4.4}$$

where Ω is a typical (constant) angular speed. The perturbation equations (2.7) and (2.8) now become

$$(DD_* - a^2)\frac{\partial u}{\partial \tau} + 2a^2\lambda\left[\frac{V_1(x)}{x}\cos\tau + \epsilon\frac{V_2(x)}{x}\sin\tau\right]v = 0 \tag{4.5}$$

and

$$\frac{\partial v}{\partial t} + 2\lambda\,[(D_* V_1)\cos\tau + \epsilon(D_* V_2)\sin\tau]\,u = 0, \tag{4.6}$$

where

$$\lambda = \Omega/\omega \tag{4.7}$$

and now

$$D \equiv \frac{\partial}{\partial x}, \quad D_* \equiv \frac{\partial}{\partial x} + \frac{1}{x}.$$

The boundary conditions (2.9) are replaced by

$$u = 0 \quad \text{on} \quad x = \eta \quad \text{and} \quad x = 1. \tag{4.8}$$

The system (4.5), (4.6) and (4.8) will be solved by a perturbation for small ϵ. We suppose the nth eigensolution of the system is expressible in the form

$$\left.\begin{aligned} u_n(x,\tau) &= e^{\epsilon c_n \tau}\,[u_n^{(0)}(x,\tau) + \epsilon u_n^{(1)}(x,\tau) + \ldots], \\ v_n(x,\tau) &= e^{\epsilon c_n \tau}\,[v_n^{(0)}(x,\tau) + \epsilon v_n^{(1)}(x,\tau) + \ldots], \end{aligned}\right\} \tag{4.9}$$

where $u_n^{(0)}$, $v_n^{(0)}$ are the eigenfunctions when $\epsilon = 0$ and are consequently periodic in τ; $u_n^{(1)}$, $v_n^{(1)}$ are supposed also to be periodic in τ; and c_n is a constant. The stability of the system is then determined by the nature of the eigenvalue c_n (except possibly at low frequencies).

The expansion (4.9) is substituted into (4.5) and (4.6), and coefficients of like powers of ϵ are equated. This leads to the following pairs of equations:

$$\left.\begin{aligned} (DD_* - a^2)\frac{\partial u_n^{(0)}}{\partial \tau} + 2a^2\lambda\frac{V_1}{x}\cos\tau\, v_n^{(0)} &= 0, \\ \frac{\partial v_n^{(0)}}{\partial \tau} + 2\lambda\,(D_* V_1)\cos\tau\, u_n^{(0)} &= 0, \end{aligned}\right\} \tag{4.10}$$

and

$$\left.\begin{aligned}(DD_* - a^2)\frac{\partial u_n^{(1)}}{\partial \tau} + 2a^2\lambda\frac{V_1}{x}\cos\tau\, v_n^{(1)} &= -c_n(DD_* - a^2)\,u_n^{(0)} \\ &\quad - 2a^2\lambda\frac{V_2}{x}\sin\tau\, v_n^{(0)}, \\ \frac{\partial v_n^{(1)}}{\partial \tau} + 2\lambda(D_*V_1)\cos\tau\, u_n^{(1)} &= -c_n v_n^{(0)} - 2\lambda(D_*V_2)\sin\tau\, u_n^{(0)},\end{aligned}\right\} \tag{4.11}$$

with $u_n^{(0)} = u_n^{(1)} = 0$ on $x = \eta$ and $x = 1$.

The equations (4.10) are of course merely those of the rigid-body oscillation and have elementary separable solutions which are conveniently written

$$\left.\begin{aligned}u_n^{(0)} &= [\alpha\cos(\sigma_n\sin\tau) + \beta\sin(\sigma_n\sin\tau)]\,\phi_n(x), \\ v_n^{(0)} &= -\frac{2\lambda(D_*V_1)}{\sigma_n}[\alpha\sin(\sigma_n\sin\tau) - \beta\cos(\sigma_n\sin\tau)]\,\phi_n(x),\end{aligned}\right\} \tag{4.12}$$

where α, β are arbitrary constants, $\phi_n(x)$ is the solution of the equation

$$\left[DD_* - a^2 + \frac{4a^2\lambda^2}{\sigma_n^2}\frac{V_1(D_*V_1)}{x}\right]\phi_n = 0 \tag{4.13}$$

with $\phi_n(\eta) = \phi_n(1) = 0$, and where Rayleigh's criterion determines whether the eigenvalue σ_n is real or imaginary.

A particular solution of the inhomogeneous system (4.11) cannot be found in separable form. We therefore use a standard approximation technique wherein the x-dependence is dealt with by expanding $u_n^{(1)}$ and $v_n^{(1)}$ in series of the eigenfunctions ϕ, the coefficients being functions of time. That is, we put

$$u_n^{(1)} = \sum_{m=1}^{\infty} A_m(\tau)\,\phi_m(x) \quad \text{and} \quad v_n^{(1)} = -2\lambda(D_*V_1)\sum_{m=1}^{\infty}\frac{1}{\sigma_m}B_m(\tau)\,\phi_m(x) \tag{4.14}$$

and substitute these, together with (4.12), into (4.11). This gives

$$\sum_{m=1}^{\infty}\frac{1}{\sigma_m^2}\left[\frac{dA_m}{d\tau} + \sigma_m\cos\tau B_m\right]\phi_m = -\frac{1}{\sigma_n^2}\phi_n\left\{c_n[\alpha\cos(\sigma_n\sin\tau) + \beta\sin(\sigma_n\sin\tau)] + \sigma_n\frac{V_2}{V_1}\sin\tau\,[\alpha\sin(\sigma_n\sin\tau) - \beta\cos(\sigma_n\sin\tau)]\right\} \tag{4.15a}$$

and

$$\sum_{m=1}^{\infty}\frac{1}{\sigma_m}\left[\frac{dB_m}{d\tau} - \sigma_m\cos\tau A_m\right]\phi_m = -\frac{1}{\sigma_n}\phi_n\left\{c_n[\alpha\sin(\sigma_n\sin\tau) - \beta\cos(\sigma_n\sin\tau)] - \sigma_n\frac{D_*V_2}{D_*V_1}\sin\tau\,[\alpha\cos(\sigma_n\sin\tau) + \beta\sin(\sigma_n\sin\tau)]\right\}. \tag{4.15b}$$

Since the functions ϕ are orthogonal in $(\eta, 1)$ with respect to the weighting factor $[V_1(D_*V_1)]^{\frac{1}{2}}$, we multiply each of the equations (4.15) by $V_1(D_*V_1)\,\phi_n$ and integrate over the range. Introducting the notation

$$p_n = \frac{\int_\eta^1 V_2(D_*V_1)\,\phi_n^2\,dx}{\int_\eta^1 V_1(D_*V_1)\,\phi_n^2\,dx} \quad \text{and} \quad q_n = \frac{\int_\eta^1 V_1(D_*V_2)\,\phi_n^2\,dx}{\int_\eta^1 V_1(D_*V_1)\,\phi_n^2\,dx}, \tag{4.16}$$

we obtain the following pair of equations:

$$\left.\begin{aligned} \frac{dA_n}{d\tau}+\sigma_n\cos\tau\,B_n &= h_{11}\alpha+h_{12}\beta,\\ \frac{dB_n}{d\tau}-\sigma_n\cos\tau\,A_n &= h_{21}\alpha+h_{22}\beta, \end{aligned}\right\} \tag{4.17}$$

where

$$\left.\begin{aligned} h_{11} &= -c_n\cos(\sigma_n\sin\tau)-\sigma_n p_n\sin\tau\sin(\sigma_n\sin\tau),\\ h_{12} &= -c_n\sin(\sigma_n\sin\tau)+\sigma_n p_n\sin\tau\cos(\sigma_n\sin\tau),\\ h_{21} &= -c_n\sin(\sigma_n\sin\tau)+\sigma_n q_n\sin\tau\cos(\sigma_n\sin\tau),\\ h_{22} &= c_n\cos(\sigma_n\sin\tau)+\sigma_n q_n\sin\tau\sin(\sigma_n\sin\tau). \end{aligned}\right\} \tag{4.18}$$

We now seek a condition that (4.17) should yield solutions A_n and B_n which are periodic with period 2π. The result can be written down from standard theory of differential equations as follows (Coddington & Levinson 1955, p. 74). If $g(\tau)$ is the matrix of coefficients of the homogeneous form of (4.17), that is if

$$g(\tau) = \begin{pmatrix} 0 & -\sigma_n\cos\tau \\ \sigma_n\cos\tau & 0 \end{pmatrix}, \tag{4.19}$$

and if $\tilde{Z}(\tau)$ is the 2×2-matrix solution of the *adjoint* to the system

$$\frac{dZ}{d\tau} = gZ, \quad Z(0) = I, \tag{4.20}$$

then (4.17) has periodic solutions if and only if

$$\int_0^{2\pi}\tilde{Z}(\tau)\,h(\tau)\,\gamma\,d\tau = 0, \tag{4.21}$$

where

$$h(\tau) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \quad\text{and}\quad \gamma = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{4.22}$$

The vector equation (4.21) is of course two equations, involving the eigenvalues c_n and the initial values α, β. For arbitrary α and β the values of c_n are the roots of the determinantal equation

$$\left|\int_0^{2\pi}\tilde{Z}(\tau)\,h(\tau)\,d\tau\right| = 0. \tag{4.23}$$

From (4.20) it is easily shown that

$$Z = \begin{pmatrix} \cos(\sigma_n\sin\tau) & \sin(\sigma_n\sin\tau) \\ -\sin(\sigma_n\sin\tau) & \cos(\sigma_n\sin\tau) \end{pmatrix} \tag{4.24}$$

so that the integrals in (4.23) can be written down immediately. Some of these integrals are identically zero, and after some simplification we obtain

$$c_n = \pm\frac{1}{2\pi}(p_n-q_n)\,\sigma_n\int_0^{2\pi}\sin\tau\sin(\sigma_n\sin\tau)\cos(\sigma_n\sin\tau)\,d\tau. \tag{4.25}$$

This remaining integral is a Bessel-function representation (cf. Watson 1966, p. 20) and so we have finally

$$c_n = \pm\tfrac{1}{2}(p_n-q_n)\,\sigma_n J_1(2\sigma_n). \tag{4.26}$$

The foregoing analysis is based on the hypothesis that the disturbance equations admit a solution of the form (4.9), together with the representation (4.14). Although not substantiated here, this type of approximation scheme is of frequent use and its validity is assumed. If this is the case, an inspection of the formula (4.26) shows that both values of the characteristic exponent c_n are *real* irrespective of whether σ_n is real or imaginary, and irrespective of the values of p_n and q_n. Consequently it appears that all flows of the type (4.3) are unstable in this sense; a phase-differential in the radial direction, however small, is sufficient to cause instability in a flow with zero mean.

This conclusion requires modification in three exceptional cases: (i) when the parameters of the flow and the wave-number are such that $2\sigma_n$ is a zero of J_1 the disturbance is marginally stable, i.e. $c_n = 0$. This is a singularity of the mathematical analysis and appears to be without any special physical significance. (ii) When $p_n = q_n$, the nth mode of the disturbance is marginally stable for all σ_n. By reference to (4.16) we see that this is in fact the condition that

$$\int_\eta^1 \phi_n^2 \left[V_1(D_* V_2) - V_2(D_* V_1)\right] dx = 0.$$

The significance of this case is discussed in §6. (iii) A further exception has to be made when the frequency of the oscillation is very small or rather when

$$\lambda = \Omega/\omega \gg 1. \tag{4.27}$$

In this case our interpretation of the result (4.26) must be modified in the light of the remarks in §3: if $V_1(r)$ is stable the phase-differential causes instability in the same way as above. If, on the other hand, $V_1(r)$ is unstable, there is again instability of the flow (4.3) but this is still the quasi-steady instability of the rigid-body oscillation, modified by, but not due to, the phase-differential.

We now briefly turn our attention to the more general zero-mean flow (4.3) in the case when ϵ is *not small.*

This will obviously be a far more complex problem than the one just considered and will involve a good deal of computation. It is not proposed to proceed with the calculations here, as the problem will be studied in detail in a subsequent paper, with the effects of viscosity taken into account.

When $\epsilon \gg 1$, however, a solution can again be obtained by a perturbation technique. If we set

$$\epsilon_1 = 1/\epsilon \quad \text{and} \quad \lambda_1 = \lambda\epsilon, \tag{4.28}$$

the disturbance equations (4.5) and (4.6) become

$$(DD_* - a^2)\frac{\partial u}{\partial \tau} + 2a^2\lambda_1\left[\frac{V_2}{x}\sin\tau + \epsilon_1\frac{V_1}{x}\cos\tau\right]v = 0,$$

$$\frac{\partial v}{\partial \tau} + 2\lambda_1[D_* V_2 \sin\tau + \epsilon_1 D_* V_1 \cos\tau]u = 0.$$

These are clearly in essence the same equations as (4.5) and (4.6) and can be treated in the same way. When $\epsilon_1 = 0$ we have stable rigid-body oscillations; and when $\epsilon_1 \neq 0$, but is small we have a phase-differential destabilization as before.

Consequently the flow (4.3) is unstable for both $\epsilon \ll 1$ and $\epsilon \gg 1$. It seems reasonable to expect therefore that it is also unstable for intermediate value of ϵ.

5. Flows with non-zero mean

In this category we consider first a profile which represents a *small* departure from steady flow. We take as a typical flow

$$\bar{v}(r,t) = V_1(r) + \epsilon V_2(r) \cos \omega t, \tag{5.1}$$

which when $\epsilon \ll 1$ lends itself to solution by a perturbation method. The disturbance equations (2.7) and (2.8) are again appropriate and, after non-dimensionalization with the scheme (4.4), they become

$$(DD_* - a^2)\frac{\partial u}{\partial \tau} + 2a^2\lambda \left[\frac{V_1}{x} + \epsilon \frac{V_2}{x} \cos \tau\right] v = 0 \tag{5.2}$$

and

$$\frac{\partial v}{\partial \tau} + 2\lambda \left[D_* V_1 + \epsilon D_* V_2 \cos \tau\right] u = 0, \tag{5.3}$$

with boundary conditions as before.

It will appear in the subsequent investigations that the effects of time-dependence in the flow (5.1) are different according as $V_1(r)$ is itself stable or unstable. It is therefore convenient to discuss the two situations separately.

(i) *When $V_1(r)$ is stable*

The eigenfunctions when $\epsilon = 0$ are known in principle and can conveniently be written

$$u = u_n^{(0)} = [\alpha \cos \sigma_n \tau + \beta \sin \sigma_n \tau]\, \phi_n(x), \tag{5.4}$$

$$v = v_n^{(0)} = -\frac{2\lambda}{\sigma_n}(D_* V_1)\,[\alpha \sin \sigma_n \tau - \beta \cos \sigma_n \tau]\phi_n(x), \tag{5.5}$$

where all the eigenvalues σ_n are real in this case, the functions $\phi_n(x)$ again satisfy (4.13), and α, β are initial constants.

For the system (5.2) and (5.3) we seek a solution which has the form

$$u_n = \exp\{\epsilon c_n \tau\}\left\{u_n^{(0)} + \epsilon \sum_{m=1}^{\infty} A_m(\tau)\, \phi_m(x) + \ldots\right\} \tag{5.6}$$

and

$$v_n = \exp\{\epsilon c_n \tau\}\left\{v_n^{(0)} - \epsilon\, 2\lambda (D_* V_1) \sum_{m=1}^{\infty} \frac{1}{\sigma_m} B_m(\tau)\, \phi_m(x) + \ldots\right\}, \tag{5.7}$$

where A_m and B_m are to be periodic functions of τ. On substituting these into (5.2) and (5.3), and equating coefficients of like powers of ϵ, we have of course that the zero-order system is identically satisfied, while the first-order system is, after multiplying by $V_1 D_* V_1 \phi_n$ and integrating,

$$\frac{dA_n}{d\tau} + \sigma_n B_n = -c_n(\alpha_n \cos \sigma_n \tau + \beta \sin \sigma_n \tau) - \sigma_n p_n \cos \tau\, (\alpha \sin \sigma_n \tau - \beta \cos \sigma_n \tau) \tag{5.8}$$

and

$$\frac{dB_n}{d\tau} - \sigma_n A_n = -c_n(\alpha \sin \sigma_n \tau - \beta \cos \sigma_n \tau) + \sigma_n q_n \cos \tau\, (\alpha \cos \sigma_n \tau + \beta \sin \sigma_n \tau), \tag{5.9}$$

where p_n and q_n are defined in (4.16).

For general values of σ_n these equations admit periodic solutions only if $c_n = 0$. In this case the eigenfunctions are periodic to $O(\epsilon)$; i.e. the disturbance is a stable one.

An exceptional case occurs when

$$\sigma_n = \tfrac{1}{2}; \tag{5.10}$$

this is a resonance phenomenon, the well-known 'subharmonic response' which is to be found in the general behaviour of equations with periodic coefficients, such as the Mathieu equation. The eigenvalue problem for c_n in this case can be treated as in §4: we require that the solutions A_n, B_n be free from secular terms $\tau \cos \frac{1}{2}\tau$, $\tau \sin \frac{1}{2}\tau$. Some straightforward algebra leads quickly to the result

$$c_n = \pm \tfrac{1}{8}(p_n - q_n) \tag{5.11}$$

as the periodicity condition. The disturbance is therefore unstable for any flows V_1, V_2 with the exception of the singular case when $p_n = q_n$.

In fact this instability occurs not only when $\sigma_n = \frac{1}{2}$, but within an ϵ-band of this value (as with the Mathieu equation); the growth rate (5.11) is however the maximum for the instabilities in this band. Also analogously with the Mathieu equation, further resonances of this type will be found to be present in the neighbourhood of $\sigma_n = 1, \frac{3}{2}, \ldots$ if the calculation is pursued to order $\epsilon^2, \epsilon^3, \ldots$ but it does not seem worthwhile to continue with the calculations here.

It is interesting to note that Kirchgässner (1960) has stated (though without proof) a stability criterion for a general flow $\bar{v}(r, t)$ in the configuration under discussion. This is that a sufficient condition for the stability of the flow $\bar{v}(r, t)$ is

$$\bar{v}(r,t) > 0, \quad D_* \bar{v} > 0$$

in $R_1 \leqslant r \leqslant R_2$, all t. The above result appears to constitute a counter-example to this criterion.

(ii) $V_1 r$ *unstable*

The resonance destabilization just discussed is clearly only relevant when σ_n is real, and so has no application to unstable disturbances. In this latter case the effect is rather a modification of the growth-rate and we seek a solution which represents this.

Let γ_n be the growth-rate ($= i\sigma_n$) and write the eigenfunctions when $\epsilon = 0$ as

$$u_n^{(0)} = \exp\{\gamma_n \tau\} \phi_n(x), \quad v_n^{(0)} = -\frac{2\lambda}{\gamma_n}(D_* V_1) \exp\{\gamma_n \tau\} \phi_n(x), \tag{5.12}$$

where $\phi_n(x)$ again satisfies (4.13), with σ_n^2 now replaced by $-\gamma_n^2$. For the perturbed eigenfunctions we circumvent some algebra by anticipating the result that the change in the growth-rate is $O(\epsilon^2)$, and so we set

$$u = \exp\{(\gamma_n + \epsilon^2 c_n)\tau\} \left\{ \phi_n(x) + \sum_{m=1}^{\infty} [\epsilon A_m(\tau) + \epsilon^2 C_m(\tau)] \phi_m(x) + \ldots \right\} \tag{5.13}$$

and

$$v = -2\lambda (D_* V_1) \exp\{(\gamma_n + \epsilon^2 c_n)\tau\} \left\{ \frac{1}{\gamma_n} \phi_n(x) + \sum_{m=1}^{\infty} \frac{1}{\gamma_m} [\epsilon B_m(\tau) + \epsilon^2 D_m(\tau)] \phi_m(x) + \ldots \right\}. \tag{5.14}$$

The procedure now is as before. We substitute (5.13) and (5.14) into the disturbance equations (5.2) and (5.3), equate coefficients of like powers of ϵ, multiply each equation by $V_1(D_* V_1)\phi_n$ and integrate with respect to x over $(\eta, 1)$. This then leads to the equations

$$\left.\begin{aligned} \frac{dA_n}{d\tau}+\gamma_n(A_n-B_n)-\gamma_n p_n\cos\tau &= 0,\\ \frac{dB_n}{d\tau}-\gamma_n(A_n-B_n)-\gamma_n q_n\cos\tau &= 0,\end{aligned}\right\} \tag{5.15}$$

as coefficients of ϵ, while the ϵ^2-system is found to be

$$\left.\begin{aligned} \frac{dC_n}{d\tau}+\gamma_n(C_n-D_n)+c_n-\gamma_n p_n\cos\tau B_n &= 0,\\ \frac{dD_n}{d\tau}-\gamma_n(C_n-D_n)+c_n-\gamma_n q_n\cos\tau A_n &= 0.\end{aligned}\right\} \tag{5.16}$$

The particular solutions of (5.15) are immediately obtained. They are

$$A_n = \frac{\gamma_n^2(p_n-q_n)}{1+4\gamma_n^2}\cos\tau+\frac{\gamma_n p_n+2\gamma_n^3(p_n+q_n)}{1+4\gamma_n^2}\sin\tau \tag{5.17}$$

and
$$B_n = \frac{\gamma_n^2(q_n-p_n)}{1+4\gamma_n^2}\cos\tau+\frac{\gamma_n q_n+2\gamma_n^3(p_n+q_n)}{1+4\gamma_n^2}\sin\tau. \tag{5.18}$$

When these are substituted into (5.16) it is clear that the solutions C_n and D_n in general contain terms proportional to τ; but these can be eliminated if c_n takes an appropriate value. Some elementary algebra leads quickly to the result

$$c_n = -\frac{\gamma_n^3(p_n-q_n)^2}{4(1+4\gamma_n^2)} \tag{5.19}$$

as the condition for absence of the secular terms.

This formula shows that for unstable $V_1(r)$, c_n is always real irrespective of the particular shape of the basic-flow velocities. Moreover, c_n is negative when γ_n is positive and vice versa, which means in particular that the positive growth-rate of an unstable disturbance is diminished by an amount $\epsilon^2 c_n$. It is this feature which has been the basis of the experimental results of Donnelly, Relf & Suhl (1962) and Donnelly (1964). They have found that the critical value of the Taylor number for a given steady flow between rotating cylinders is increased when a small amplitude oscillatory flow is superimposed. Although the actual increment in the Taylor number can only be found by including viscosity, the above calculations clearly indicate a trend towards stabilization of unstable disturbances. On the other hand (5.19) suggests a destabilization of the conjugate, damped disturbances. This effect has not been observed experimentally, and presumably it will be eliminated by viscous action. But it follows that a complete explanation of Donnelly's observations can only be provided by treating the full viscous problem.

The result (5.19) immediately raises the question of whether all growing disturbances can be completely stabilized by taking ϵ large enough. It is therefore particularly interesting to examine the profile (5.1) when ϵ is *not small*.

The result can be obtained directly when $\epsilon \gg 1$. In the limit $\epsilon \to \infty$ the flow becomes the zero-mean rigid-body oscillation $V_2(r)\cos\omega t$, which is stable. When ϵ is very large but finite we are in effect considering the profile

$$\bar{v}(r,t) = V_2(r)\cos\omega t + \epsilon_1 V_1(r), \tag{5.20}$$

with $\epsilon_1 = 1/\epsilon \ll 1$, the disturbance equations (5.2) and (5.3) being appropriately modified. The oscillatory component of the flow is now dominant and it is not difficult to show, by a technique similar to that used in § 4, that (5.20) is in general unstable for every particular $V_1(r)$ and $V_2(r)$. It follows that even if (5.1) is completely stabilized as ϵ increases, it becomes unstable again when ϵ is sufficiently large. This behaviour was also found by Conrad & Criminale (1965*b*).

When $\epsilon = O(1)$ we are again faced with a difficult computational problem which will not be tackled here in any generality, since quantitative results in the absence of viscosity are not very meaningful. However, insight into the behaviour of the disturbances can be gained by considering a special, simplifying case of (5.1).

We take

$$\bar{v}(r,t) = V_1(r) + (\epsilon/r)\cos\omega t, \tag{5.21}$$

where $V_1(r)$ is an unstable flow and ϵ is not necessarily small. Since now

$$D_*(1/r) = 0$$

the equations (5.2) and (5.3) are at once reducible to the single equation

$$(DD_* - a^2)\frac{\partial^2 v}{\partial \tau^2} - 4a^2\lambda^2 D_* V_1\left[\frac{V_1}{x} + \frac{\epsilon}{x^2}\cos\tau\right]v = 0, \tag{5.22}$$

in which v has been replaced by $(D_* V_1)\,v$. To solve this we put

$$v = \sum_{m=1}^{\infty} A_m(\tau)\,\phi_m(x), \tag{5.23}$$

where the $\phi_m(x)$ are the eigenfunctions of the flow V_1, to each of which corresponds an eigenvalue γ_m, at least one of which is real; and the $A_m(\tau)$ are to be determined. We next follow the usual orthogonalization procedure, and this leads to the infinite set of equations

$$\frac{d^2 A_n}{d\tau^2} - \gamma_n^2 A_n - \epsilon\gamma_n^2 \sum_{m=1}^{\infty} p_{nm} A_m \cos\tau = 0, \tag{5.24}$$

with $n = 1, 2, \ldots$ and

$$p_{nm} = \frac{\displaystyle\int_\eta^1 \frac{1}{x}(D_* V_1)\,\phi_n\phi_m\,dx}{\displaystyle\int_\eta^1 V_1(D_* V_1)\,\phi_n^2\,dx}.$$

The extent to which this system can be simplified and information inferred about its solutions depends on the importance of the coupling between the equations. If the coupling is strong there appears to be no alternative to a formidable computational problem. On the other hand, if the coupling is zero each member of (5.24) reduces to a Mathieu equation, the properties of which are well known.

Therefore when the coupling is weak it seems reasonable to suppose that the principal features of the solutions are similar to those of Mathieu functions.

If the γ_n are arranged in decreasing order of magnitude then the function A_1 will be given approximately by the equation

$$\frac{d^2A_1}{d\tau^2} - \gamma_1^2[1+\epsilon p_{11}\cos\tau]A_1 = 0 \tag{5.25}$$

provided $p_{11} \gg p_{12} \gg p_{13} \ldots$. This inequality will in general hold if the function $V_1(r)$ is such that the eigenvalues γ_n are moderately widely spaced, and in such circumstances (5.25) will describe the dominant behaviour of the instability.

Assuming these conditions, we compare (5.25) with the standard Mathieu equation

$$\frac{d^2y}{dz^2} + [a \pm 2q\cos 2z]y = 0 \tag{5.26}$$

and examine solutions for fixed $a < 0$. It is known (see, for example, McLachlan 1964, p. 41) that, as q increases from zero, the solution which is unstable with growth-rate $(-a)^{\frac{1}{2}}$ when $q = 0$ becomes monotonically less unstable until it reaches a point of neutral stability when $q = -a$ approximately. Then, with q continuing to increase, there is a narrow band in which all solutions are stable, followed again by instability. Further stable regions occur as $q \to \infty$, but they become increasingly narrow.

Comparison with (5.25) shows that for fixed γ_1 and increasing ϵ, the solution A_1 is mainly unstable except within certain stable bands, the first and widest of which occurs when

$$\epsilon \approx 2/p_{11}. \tag{5.27}$$

As ϵ increases from zero to the value $2/p_{11}$ the growth-rate decreases monotonically from γ_1 to zero.

Certainly this conclusion must be modified in the presence of finite coupling between the equations (5.24), and it may be that the stability region is displaced, reduced in width or even eliminated altogether. Also it should be noted that we have been considering only the lowest mode of the uncoupled system and that even when this is stable the second and higher modes may continue to be unstable. Nevertheless, the above results suggest that there is likely to be an optimum value of ϵ, of order 1, at which maximum stabilization occurs, particularly in an real fluid when viscosity may be expected to damp out all except the lowest mode.

Finally it may be pointed out that the behaviour-patterns discussed in this section continue in general to apply when: (*a*) the profile (5.1) is generalized to

$$\bar{v}(r,t) = V_1(r) + \epsilon[V_2(r)\cos\omega t + V_3(r)\sin\omega t],$$

i.e. with a phase-differential in the oscillatory component of the flow; (*b*) the frequency ω is very small, subject to the qualifications of §3; (*c*) the profile has the form

$$\bar{v}(r,t) = V_1(r)[1+\cos\omega t] + \epsilon[V_2(r)\cos\omega t + V_3(r)\sin\omega t].$$

Still more general flows might be expected to behave in a generally similar manner.

6. Discussion

The results of the foregoing analyses may be summarized as follows.

(i) Flows which are periodic about a zero mean are in general unstable, except for rigid-boundary oscillations of the fluid. This certainly appears to be the case for both small and large departures from the rigid-body state, and the nature of the equations strongly suggests that it will also be the case in the intermediate region. A further, interesting exception has been noted, namely that for an order-ϵ perturbation to the nth mode of the rigid-body solution, a neutrally-stable disturbance occurs when

$$\int_{\eta}^{1} \phi_n^2 \left[V_1(D_* V_2) - V_2(D_* V_1) \right] dx = 0. \tag{6.1}$$

It may be inferred from this that *all* the modes will remain neutrally stable only if

$$V_1(D_* V_2) - V_2(D_* V_1) = 0. \tag{6.2}$$

For the flow in question, given by (4.1), equation (6.2) is simply the condition that the vorticity of the flow and its velocity are everywhere in phase. The criterion which this suggests, therefore, is that instability occurs when velocity and vorticity are out of phase. This is a new type of instability, associated with the periodic motion and different from, and independent of, inflexion-point or centrifugal instabilities.

Some inferences can now be drawn concerning the role of viscosity. In the first place the phase-difference between velocity and vorticity is a direct consequence of viscous action. This is evidenced by any solution of the diffusion equation having time-periodic boundary conditions, of which the classical Stokes shear-layer is a prototype. It is only in the absence of viscosity that a basic-flow solution having the property (6.2) can be found. Thus, although the above instability has been found from the non-viscous disturbance equations, it may be said that viscosity is really the cause of it in that it gives rise to the unstable basic flow.

In addition to this, viscosity will have an influence on the disturbances themselves, and it is anticipated that its effect will be able to damp the growth-rates and possibly to stabilize the flow completely in some subcritical region. The details will be examined in a subsequent paper, but there seems no reason to modify the above conclusions in a general sense.

(ii) When the flow is an oscillation about a non-zero mean, the oscillatory component destabilizes a stable mean flow and tends to stabilize an unstable one. Equations (5.11) and (5.19) suggest that exceptions occur when the velocity and vorticity are in phase, from which we might infer that a situation obtains similar to that discussed above.

These conclusions refer to situations where the oscillatory component is not too large in amplitude compared with the mean steady component; for if it is too large, the flow will more nearly resemble the zero-mean flow of (i) above. In the presence of viscosity, neutrally-stable inviscid disturbances are damped so that destabilization, while continuing to be present as a tendency, will only actually occur when the amplitude of the oscillatory component is large enough to overcome the viscous damping.

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